

# ON THE REPRESENTATION OF A POLYNOMIAL AS A SUM OF MONOMIALS

MILOŠ ARSENOVIĆ AND RADOŠ BAKIĆ

ABSTRACT. We investigate systems of  $m \geq n$  polynomial equations in  $n$  complex variables, where polynomials involved are power sums, and give estimates of the solutions. These results are applied to obtain a representation of arbitrary polynomial of degree  $n$  as the average of powers  $(z - z_j)^n$ .

Two basic families of symmetric polynomials in  $n$  variables  $x_1, \dots, x_n$  are elementary symmetric polynomials  $\sigma_k = \sigma_k(x_1, \dots, x_n)$  and power sums  $s_k = s_k(x_1, \dots, x_n)$  defined by

$$\sigma_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}, \quad 1 \leq k \leq n,$$

$$\sigma_k = 0, \quad k > n,$$

and

$$s_k = x_1^k + \dots + x_n^k, \quad k \geq 1.$$

These families are related by well-known identities:

$$(1) \quad k\sigma_k = \sum_{i=1}^k (-1)^{i-1} s_i \sigma_{k-i}, \quad k \geq 1.$$

These relations are linear with respect to both families  $\sigma_j$  and  $s_k$ , hence we can express members of one family in terms of members of the other family. Explicitly

$$(2) \quad \sigma_k = \frac{1}{k!} \begin{vmatrix} s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ s_{k-1} & s_{k-2} & s_{k-3} & \dots & k-1 \\ s_k & s_{k-1} & s_{k-2} & \dots & s_1 \end{vmatrix} = \Delta_k(s_1, \dots, s_k)$$

and

$$(3) \quad s_k = \begin{vmatrix} \sigma_1 & 1 & 0 & \dots & 0 \\ 2\sigma_2 & \sigma_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (k-1)\sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \dots & 1 \\ k\sigma_k & \sigma_{k-1} & \sigma_{k-2} & \dots & \sigma_1 \end{vmatrix} = D_k(\sigma_1, \dots, \sigma_k).$$

Let us note that equations (2) and (3) are equivalent, both being equivalent to (1).

We say that a monomial  $As_1^{a_1} \dots s_k^{a_k}$  is  $k$ -regular in  $s_1, \dots, s_k$  if  $a_1 + 2a_2 + \dots + ka_k = k$  and  $A$  is a real number such that  $\operatorname{sgn} A = (-1)^{k+a_1+\dots+a_k}$ .

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**Lemma 1.** *For any  $k \geq 1$  polynomial  $\sigma_k$  can be expressed as a sum of  $k$ -regular monomials in  $s_1, \dots, s_k$ .*

*Proof.* We use induction on  $k$  and relations (1). For  $k = 1$  we have  $\sigma_1 = s_1$  as desired. Now we assume the statement is true for  $1 \leq j < k$ . Because of (1) and induction hypothesis we need only to prove that if  $As_1^{a_1} \dots s_{k-i}^{a_{k-i}}$  is  $(k-i)$ -regular then  $(-1)^{i-1} s_i As_1^{a_1} \dots s_{k-i}^{a_{k-i}}$  is  $k$ -regular, which can be easily checked.  $\square$

Our estimates of the moduli of solutions of systems of polynomial equations are based on the following theorem due to Tchakaloff, see [3].

**Theorem 1.** *Let  $f_i(z)$  be a finite set of complex polynomials of degree  $n$ , all having a positive leading coefficient. If all  $f_i(z)$  have their zeros in the disk  $|z - z_0| \leq r$ , then polynomial  $F(z) = \sum_i f_i(z)$  has all its zeros in the disk  $|z - z_0| \leq C_n r$ , where*

$$(4) \quad C_n = \frac{1}{\sin \frac{\pi}{2n}}.$$

Further results related to the above theorem can be found in [2].

**Theorem 2.** *Suppose that the following system in variables  $z_1, \dots, z_n$  is given*

$$(5) \quad z_1^j + z_2^j + \dots + z_n^j = b_j, \quad 1 \leq j \leq n.$$

*Then it has unique solutions for  $z_1, \dots, z_n$  (up to permutations). Also  $|z_i| \leq C_n M$  where  $C_n$  is the constant given in (4) and  $M = \max_{1 \leq j \leq n} |b_j|^{1/j}$ .*

*Proof.* Our system can be written in the form  $s_j(z_1, \dots, z_n) = b_j$ ,  $1 \leq j \leq n$ . Using (2) and (3), we can rewrite it in an equivalent form

$$(6) \quad \sigma_k(z_1, \dots, z_n) = c_k, \quad 1 \leq k \leq n,$$

where  $c_k = \Delta_k(b_1, \dots, b_k)$ . By the Fundamental theorem of algebra and Viet's rule, the above system has unique solution for  $z_j$  (up to permutations). Since two previous systems are equivalent, the first part of our lemma is proved. Let  $(z_1, \dots, z_n)$  be a solution; it suffices, by symmetry, to prove  $|z_n| \leq C_n M$ . Let  $s'_k = z_1^k + \dots + z_{n-1}^k$  be power sum polynomials in  $z_1, \dots, z_{n-1}$ , and let  $\sigma'_k$  be the corresponding elementary symmetric polynomials. By the above lemma, it follows that  $\sigma'_n$  can be expressed as a sum of  $n$ -regular monomials  $P_\lambda$ ,  $1 \leq \lambda \leq l$  in  $s'_1, \dots, s'_l$ . Let  $A_\lambda \prod_{j=1}^n (s'_j)^{a_{j,\lambda}}$  be one of them.

Using (5) we get  $s'_j = b_j - z_n^j$ , hence we can rewrite previous monomial as  $A_\lambda \prod_{j=1}^n (b_j - z_n^j)^{a_{j,\lambda}}$ . Hence, since our monomial is  $n$ -regular, it can be seen as a polynomial  $P_\lambda(z_n)$  in  $z_n$  of degree  $n$ , the sign of the leading coefficient is equal to  $(-1)^{a_{1,\lambda} + \dots + a_{n,\lambda}} \text{sgn} A_\lambda = (-1)^n$ . However,  $\sigma'_n$  is identically equal to zero, hence we have equation  $\sum_{\lambda=1}^l P_\lambda(z_n) = 0$  where  $\deg P_\lambda = n$  for all  $1 \leq \lambda \leq l$  and all the leading coefficients of  $P_\lambda$  have the same sign  $(-1)^n$ . Note that all such polynomials have all their zeros in the disc  $|z| \leq M$ . Hence by Theorem 1 their sum has all its zeros in the disc  $|z| \leq C_n M$ , therefore  $|z_n| \leq MC_n$  as required.  $\square$

**Theorem 3.** *Any complex polynomial*

$$(7) \quad f(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$$

*can be written in the form*

$$(8) \quad f(z) = \frac{1}{n} \sum_{i=1}^n (z - z_i)^n,$$

where  $z_1, \dots, z_n$  are uniquely determined up to their order. Also, if  $M = \max_{1 \leq j \leq n} |c_j|^{1/j}$ , where  $c_j = \frac{n a_{n-j}}{\binom{n}{j}}$  then we have an estimate  $|z_i| \leq C_n M$ , where  $C_n$  is the constant given in (4).

*Proof.* Relation (8) leads to the following system of equations:

$$(9) \quad z_1^j + \dots + z_n^j = (-1)^j c_j, \quad 1 \leq j \leq n.$$

According to Theorem 1 above system has desired solution which proves the theorem.  $\square$

Next, motivated by Theorem 2, we consider the following system in  $m$  complex variables, where  $m \geq n$ :

$$(10) \quad \frac{1}{m}(z_1^j + \dots + z_m^j) = A_j^j, \quad 1 \leq j \leq n.$$

Since  $m \geq n$ , it follows from Theorem 2 that a solution exists. We are interested in estimates of solutions. A general problem is to find the best possible constant  $K_{n,m}$  such that for any complex numbers  $A_1, \dots, A_n$  there is a solution to the system (10) satisfying  $\max_{1 \leq k \leq m} |z_k| \leq K_{n,m} A$ , where  $A = \max_{1 \leq j \leq n} |A_j|$ . A weaker version of the same problem is to find  $K_n = \inf_{m \geq n} K_{n,m}$ . An asymptotic behavior of  $K_n$  would be of interest as well.

Our last theorem gives a partial answer to the questions posed above. Let us define a sequence  $D_n$  inductively by  $D_1 = 1$  and

$$(11) \quad D_{n+1} = 1 + (1 + D_n^{n+1})^{1/n+1}, \quad n \geq 1.$$

**Theorem 4.** *If  $m = n!$  then the system (10) has a solution  $(z_k)_{1 \leq k \leq n!}$  such that*

$$(12) \quad \max_{1 \leq k \leq n!} |z_k| \leq D_n \max_{1 \leq j \leq n} |A_j|.$$

*Proof.* We proceed by induction on  $n$ . The case  $n = 1$  is trivial, so we assume the statement is true for  $n$ . Let  $A_1, \dots, A_n, A_{n+1}$  be arbitrary complex numbers and set  $A = \max_{1 \leq j \leq n+1} |A_j|$ . By inductive hypothesis there exist complex numbers  $z_k$ ,  $1 \leq k \leq n!$  such that

$$(13) \quad \frac{1}{n!} \sum_{k=1}^{n!} z_k^j = A_j^j, \quad 1 \leq j \leq n$$

and

$$(14) \quad \max_{1 \leq k \leq n!} |z_k| \leq D_n \max_{1 \leq j \leq n} |A_j| \leq D_n A.$$

Set  $\xi = \exp \frac{2\pi i}{n+1}$  and define  $(n+1)!$  complex numbers

$$(15) \quad w_{k,l} = z_k + \xi^l K, \quad 1 \leq k \leq n!, \quad 0 \leq l \leq n$$

where constant  $K$  will be chosen later on. Using elementary identity

$$\sum_{l=0}^n \xi^{lr} = \begin{cases} n+1 & \text{if } n+1 \text{ divides } r \\ 0 & \text{otherwise} \end{cases}$$

we deduce that for every  $1 \leq j \leq n$  and  $1 \leq k \leq n!$  we have

$$\begin{aligned} \sum_{l=0}^n w_{k,l}^j &= \sum_{l=0}^n (z_k + \xi^l K)^j = \sum_{l=0}^n \sum_{r=0}^j \binom{j}{r} z_k^{j-r} K^r \xi^{lr} \\ &= \sum_{r=0}^n \binom{j}{r} z_k^{j-r} K^r \sum_{l=0}^n \xi^{lr} = (n+1) z_k^j \end{aligned}$$

Therefore, using (13) we have

$$\frac{1}{(n+1)!} \sum_{k=1}^{n!} \sum_{l=0}^n w_{k,l}^j = \frac{1}{n!} \sum_{k=1}^n z_k^j = A_j^j$$

for every  $1 \leq j \leq n$ , which means that the  $(n+1)!$  numbers  $w_{k,l}$  satisfy the first  $n$  equations of the system (10), with  $m = (n+1)!$ , *regardless* of the choice of the constant  $K$ . Next we have

$$\begin{aligned} \frac{1}{(n+1)!} \sum_{k,l}^{n+1} w_{k,l}^{n+1} &= \frac{1}{(n+1)!} \sum_{k=1}^{n!} \sum_{l=0}^n (z_k + \xi^l K)^{n+1} \\ &= \frac{1}{(n+1)!} \sum_{k=1}^{n!} \sum_{l=0}^n \sum_{r=0}^{n+1} \binom{n+1}{r} z_k^{n+1-r} K^r \xi^{lr} \\ &= \frac{1}{(n+1)!} \sum_{k=1}^{n!} \sum_{r=0}^{n+1} \binom{n+1}{r} z_k^{n+1-r} K^r \sum_{l=0}^n \xi^{lr} \\ &= \frac{1}{n!} \sum_{k=1}^{n!} (z_k^{n+1} + K^{n+1}) \\ &= K^{n+1} + \frac{1}{n!} \sum_{k=1}^{n!} z_k^{n+1}, \end{aligned}$$

and therefore we choose  $K$  so that  $K^{n+1} = A_{n+1}^{n+1} - \frac{1}{n!} \sum_{k=1}^{n!} z_k^{n+1}$ . Using (14) we deduce  $|K|^{n+1} \leq A^{n+1} + D_n^{n+1} A^{n+1}$  and this gives  $|K| \leq A(1 + D_n^{n+1})^{1/n+1}$ . Finally we get

$$|w_{k,l}| = |z_k + \xi^l K| \leq |z_k| + |K| \leq A + A(1 + D_n^{n+1})^{1/n+1} = D_{n+1} A$$

which completes the proof.  $\square$

By the above theorem we have  $K_n \leq K_{n,n!} \leq D_n$ . Let us prove that

$$(16) \quad \lim_{n \rightarrow \infty} \frac{D_n}{n} = 1.$$

First, we have  $D_{n+1} \geq 1 + D_n$  which gives  $D_n \geq n$  for all  $n \in \mathbb{N}$ . Next

$$D_n = 1 + D_{n-1}(1 + D_{n-1}^{-n})^{1/n} \leq 1 + D_{n-1} \left( 1 + \frac{1}{n D_{n-1}^n} \right),$$

which implies

$$D_n - D_{n-1} \leq 1 + \frac{1}{n D_{n-1}^n}.$$

Since  $\lim_{n \rightarrow \infty} D_n = +\infty$  we get, using Stolz Theorem on sequences

$$\limsup_{n \rightarrow \infty} \frac{D_n}{n} \leq \limsup_{n \rightarrow \infty} (D_n - D_{n-1}) \leq \limsup_{n \rightarrow \infty} \left( 1 + \frac{1}{nD_{n-1}} \right) = 1,$$

which, combined with  $D_n \geq n$ , gives (16).

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#### REFERENCES

- [1] V, Prasolov, *Polynomials*, Springer-Verlag, 2004.
- [2] Z, Rubinstein, *Some results in the location of zeroes of polynomials*, Pacific J. of Mathematics Vol. 15, No. 4, 1965.
- [3] L. Tchakaloff, *Sur la distribution des zeros d'une classe des polynomes algebriques*, C. R. Acad. Bulgare Sci. 13 (1960), 249-251.

FACULTY OF MATHEMATICS, UNIVERSITY OF BELGRADE, STUDENTSKI TRG 16, 11000 BELGRADE, SERBIA

*E-mail address:* arsenovic@matf.bg.ac.rs

UČITELJSKI FAKULTET, BEOGRAD, SERBIA

*E-mail address:* bakicr@gmail.com